Answers to Final Exam questions

1. An agent wishes to

$$\max_x f(x)$$

subject to:

$$p'x = Y$$

where x is a vector of goods, and p is a vector of prices comformable to x. Let Y denote the agent's total income. Computing a first-order necessary condition for an optimum yields:

$$\frac{\partial f(x)}{\partial x} - \lambda p = 0$$

Let x^* denote the vector of optimal choices of x. Consider a deviation $x^+ = x^* + \Delta x$ such that

$$p'x^+ = Y$$

In other words

$$p'\Delta x = 0$$

So what is the cost in utility from deviating for the optimum? Take a Taylor approximation.

$$\begin{aligned} f(x^+) - f(x^*) &\approx \frac{\partial f(x)'}{\partial x}\Delta x + \frac{1}{2}\Delta x' \frac{\partial^2 f(x)}{\partial x^2}\Delta x \\ &\approx \lambda p'\Delta x + \frac{1}{2}\Delta x' \frac{\partial^2 f(x)}{\partial x^2}\Delta x \\ &\approx \frac{1}{2}\Delta x' \frac{\partial^2 f(x)}{\partial x^2}\Delta x \end{aligned}$$

Thus a first-order deviation in the decision rule causes a second-order cost in utility.

"Second order" does not necessarily mean small. It depends on the curvature of the utility function and the size of Δx. If the utility function is steeply curved around the optimum, so that utility falls quickly as one moves away from the optimum, then these second-order costs could be large. Also if the deviations away from the optimum are large, these costs could be large. But in the models we have studied in class with conventional functional forms (e.g. quadratic utility with linear laws of motions for the state variables), even relatively large deviations from the optimum result in relatively trivial utility costs.

One last note: Of course, a suboptimal decision that costs a large corporation one-tenth of one percent of its profits is a small mistake from the firm's perspective, but quite large from the perspective of a manager if s/he can improve on the decision and capture some of the increased profit for him/herself.

2. A no-Ponzi condition is a constraint on agents that prevents them from letting their borrowing grow indefinitely at the interest rate. Without such a condition, the agent can always finance arbitrarily large expenditures without ever repaying, simply borrowing more to repay old loans. A "Ponzi scheme" is a form of fraud in which this type of behavior is attempted. A transversality condition is a necessary and/or sufficient condition for optimality that relates to "terminal conditions" in a dynamic model. In a typical finite-horizon model of wealth accumulation it will require that all wealth be "used up" by the end of the planning horizon. A similar requirement is ordinarily needed in infinite-horizon models, but is harder to make precise. In an economic growth model it usually takes the form of requiring that wealth not grow unboundedly large or not grow as fast as the discount rate. Both no-Ponzi conditions and TVC's limit the rate of growth of wealth, but they are quite distinct. The no-Ponzi condition prevents wealth from growing more and more negative at an exponential rate, while in a growth model the TVC prevents wealth from growing more and more positive at an exponential rate. Also, a no-Ponzi constraint is perceived by the agent as imposed from outside, as a legal requirement or as a rule imposed by creditors. A TVC is imposed by an agent on his own behavior in order to maximize his objective function.

3. (a) (i) A *competitive rational expectations equilibrium* is a triple (U, y, x) such that the Phillips curve is satisfied and $x = y$.

   (ii) The *Ramsey problem* is:

$$\max_y -.5[(\bar{U} - \theta(y - x))^2 + \gamma y^2]$$

   subject to:

$$y = x.$$

   (iii) The *Ramsey outcome* is the inflation rate y that solves the Ramsey problem.

   (iv) A *best response function for the government* is

$$y = \arg\max_y -.5[(\bar{U} - \theta(y - x))^2 + \gamma y^2].$$

   Following the notes and Sargent's Conquest manuscript, we denote the government's best response by $y = B(x)$.

   (v) A *Nash equilibrium* is a pair (x, y) such that $x = y$ and $y = B(x)$.

   (b) First we will do Ramsey. The government knows that the household will observe its choice of y and thus the household will set $x = y$. Taking this into account the government wishes to:

$$\max_y -.5[(\bar{U} - \theta(y - x))^2 + \gamma y^2]$$

   subject to:

$$y = x.$$

If we substitute the constraint into the government's objective function, the government's problem becomes:

$$\max_y -.5[\bar{U}^2 + \gamma y^2]$$

The solution to this problem is to set $y = 0$. Therefore at the Ramsey outcome $x = y = 0$, and the unemployment rate is equal to the natural rate.
Now let's solve for the Nash equilibrium. The household chooses x. The government takes x as given (fixed) and wishes to:

$$\max_y -.5[(\bar{U} - \theta(y - x))^2 + \gamma y^2]$$

Setting the first derivative equal to zero yields:

$$-.5[2(\bar{U} - \theta(y - x))(-\theta) + 2\gamma y] = 0$$

We do a little algebra ...

$$\gamma y = \theta(\bar{U} - \theta(y - x))$$
$$\gamma y = \theta\bar{U} - \theta^2 y + \theta^2 x$$
$$(\gamma + \theta^2)y = \theta\bar{U} + \theta^2 x$$

And we get the government's best response is:

$$y = \frac{\theta}{\gamma + \theta^2}\bar{U} + \frac{\theta^2}{\gamma + \theta^2}x$$

In a competitive equilibrium we know

$$x = y$$

so

$$y = \frac{\theta}{\gamma + \theta^2}\bar{U} + \frac{\theta^2}{\gamma + \theta^2}y.$$

Doing a little more algebra leads to

$$\left(1 - \frac{\theta^2}{\gamma + \theta^2}\right) y = \frac{\theta}{1 + \theta^2}\bar{U}$$

So in the Nash equilibrium, $x = y = \frac{\theta}{\gamma}\bar{U}$ and $U = \bar{U}$.
In the Ramsey outcome, the government's distaste for inflation, γ, has no effect on the inflation rate. The government always sets the inflation rate to 0.
In the Nash equilibrium, the stronger the government's distaste for inflation (the larger is γ), the lower is the inflation rate. As γ gets approaches infinity, the Nash equilibrium approachs the Ramsey outcome.

4. (a) It is a menu cost model because it implies that changes in prices absorb output. The term $\xi\big((P_t - P_{t-1})/\bar{P}_t\big)^2$ represents the cost of changing prices, and is therefore the one that makes it a menu cost model. The fact that it is a monopolistic competition model is the fact that in it prices are decision variables for firms, who see a possibility (not realized in equilibrium) that they could make the price of their own product deviate from that of the average firm. Each firm has a demand curve, represented by equation (7) on the exam, which relates production relative to the average level of production $(L_t^\alpha/\bar{L}_t^\alpha)$ to price relative to the average price $(P_t/\bar{P}_t)$. As $\theta \to +\infty$ the demand curve flattens and approaches the competitive case of perfectly elastic demand.

   (b) The parameter is ξ. You should have given five minutes of explanation of this answer, though, and this might have been easiest after you had derived the FOC's. With $\xi \neq 0$, more variable time paths of prices shrink the feasible set of consumption paths, so it is clear that with $\xi \neq 0$ neutrality does not hold. That it does hold with $\xi = 0$ requires a more complicated argument: With $\xi = 0$, the FOC's and constraints can be broken into one set that involves only real variables (with real wage, w_t and real interest rate included in the real variables), not including τ, and another set consisting of the government budget constraint, fiscal policy, and monetary policy, that involves also B, P and τ. The former block can determine the real equilibrium by itself, while the latter block, unless it implies infeasibility or violation of private-sector transversality, can modify the time paths of P, τ and B arbitrarily without changing the real allocation. Note in particular that the monopolistic competition component of the model is retained, without producing any non-neutrality.

   (c) In this model all liabilities of the government bear interest, but the price level is still the rate at which they trade for goods. Since government liabilities do not provide transactions services, they are held only for the return they provide, which in turn is made possible by the government's power to tax. If the government has the power and will to increase taxes devoted to debt service as it increases outstanding debt, it can do so without raising the price level. If instead it increases the volume of its outstanding nominal debt without increasing taxes, the value of the debt in terms of real commodities (the inverse of the price level) will decline, i.e. there will be inflation.

   (d) The consumer Euler equations are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t \tag{1}$$

$$\partial L: \qquad \frac{1}{1-L} = \lambda_t w_t \tag{2}$$

$$\partial B: \qquad \frac{\lambda_t}{P_t} = \beta R_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] + \mu_t\ . \tag{3}$$

The firm Euler equations are

$$\partial\pi: \qquad \Phi_t = \zeta_t \tag{4}$$

$$\partial L: \qquad \zeta_t \left(\frac{P_t}{\bar{P}_t} \alpha L_t^{\alpha-1} - w_t \right) = \alpha \nu_t \frac{L_t^{\alpha-1}}{\bar{L}_t^{\alpha}} \tag{5}$$

$$\partial P: \qquad \zeta_t \frac{L_t^{\alpha}}{\bar{P}_t} - 2\xi\zeta_t \frac{P_t - P_{t-1}}{\bar{P}_{t-1}^2} + 2\beta\xi E_t \left[\zeta_{t+1} \frac{P_{t+1} - P_t}{\bar{P}_t^2} \right] = \nu_t \theta \frac{P_t^{-\theta-1}}{\bar{P}_t^{-\theta}} \,. \tag{6}$$

The new variables λ_t, μ_t, ζ_t, and ν_t are Lagrange multipliers, on the consumer budget constraint, the consumer $B > 0$ constraint, the firm budget constraint, and the firm demand curve, respectively.

To use these equations to answer (4b), apply the equilibrium conditions that barred variables must match their individual-firm counterparts, multiply (5) by L and (6) by $\Phi_t P_t$, and solve to eliminate Lagrange multipliers, producing

$$\frac{C_t}{1 - L_t} = w_t \tag{7}$$

$$\frac{1}{C_t P_t} = \beta R_t E_t \left[\frac{1}{P_{t+1} C_{t+1}} \right] \tag{8}$$

$$-2\xi \frac{P_t - P_{t-1}}{P_{t-1}} \frac{P_t}{P_{t-1}} + 2\beta\xi E_t \left[\frac{\Phi_{t+1}}{\Phi_t} \frac{P_{t+1} - P_t}{P_t} \right] = (\theta - 1) L_t^{\alpha} - \frac{\theta}{\alpha} w_t L_t \,. \tag{9}$$

With $\xi = 0$, (7) and (9) both involve only real variables and involve no lags or expectations. They are in terms of C, L, and w alone. They can be combined with the social resource constraint, which with $\xi = 0$ is just $C_t = L_t^{\alpha}$, to produce a static, 3-equation system in three variables that determines constant values for them. The rest of the system then has no effect on the values of these real variables.

(e) With the log utility function that has been specified, equation (8) on the exam specifies that the discount factor firms use to value future profits (Φ_{t+1}/Φ_t in (9) on the exam) is the same as that used by consumers (in (8) on the exam) to value to value future returns on their bond investments. If this condition did not hold, then an entrepreneur could create a security with a pattern of next-period payoffs that would be valued differently by firms and consumers. This would create an arbitrage opportunity. The absence of any possibility of creating a security with a pattern of payouts that is valued differently by different agents defines a complete market, and it is equivalent to saying that all agents use the same stochastic discount factor. Note that here, because of the monopolistic competition, there is no presumption that market equilibria are optimal, and thus no presumption that it is socially desirable that there be complete markets.

5. This question asks you to check existence and uniqueness by the usual root-counting criteria. Sadly, I realized as I prepared this answer that the roots given

in the problem statement do not correspond to the problem 4 model as claimed. This does not affect the application of the root-counting criteria as described below and should not have affected anyone's ability to answer the question, unless you attempted to derive the linearized model and find its roots, a task you were not asked to undertake. See the note below for a discussion of what the problem would have looked like with the correct roots.

(a) The model has two endogenous error terms. Therefore, so long as the model does not have a special structure that creates singularities, we expect to find that we need one unstable root to $\Gamma_0^{-1}\Gamma_1$ for each endogenous error term, i.e. two. You were told to consider any root larger than one in absolute value as "unstable". So the first parameter setting, which was claimed to produce roots of 1.0500, 1.0000, -0.4862, -2.1595, implies existence of a unique equilibrium and the second one, with roots of 1.0500, 1.0000, -0.1489+1.0138i, -0.1489-1.0138i, implies non-existence because it has three roots exceeding one in absolute value. The absolute value of the two imaginary roots is 1.036.

(b) Now the first parameter setting is claimed to produce only one unstable root, while the second produces two. This implies non-uniqueness for the first setting and existence with uniqueness for the second setting.

(c) The smaller system, ignoring B and the government budget constraint, is giving the wrong answers. This gets to the main point of the FTPL literature: ignoring the GBC leads to mistaken conclusions. For the first parameter setting, the apparent non-uniqueness disappears when we realize that all but one of the potential equilibria imply explosive behavior for the omitted B/P variable and would thus violate transversality. For the second parameter setting, the equilibrium that apparently exists does not in fact, because it would imply explosive behavior of government debt that violates transversality.

Note on what the problem would have looked like with correct roots:

My own suspicions that there was a mistake were aroused by the fact that there were large negative or imaginary-with-small-real-part roots. This means that the dynamics are "fast" relative to the time unit, and usually we set up discrete time models so that the dynamics are "slow" relative to the time unit. It is a good general rule, which I should have paid more attention to here, that when your model gives roots implying rapid oscillations, either you have made a mistake or there is something odd in your model's formulation. In the model of problem 4, with the stated interest rate and tax policies (keeping them constant) existence and uniqueness hold for any positive value of ξ and there are no negative or imaginary roots. If the roots had been calculated correctly they would have been 0.1349, 1.0000, 7.7848, and 1.0500 for the $\xi = .1$ case and 0.2204, 1.0000, 4.7645, 1.0500 for the $\xi = .2$ case. As before, the roots of 1.05 disappear when the government budget constraint and B are dropped from the model. Thus with the correct roots there is existence and uniqueness for both values of ξ, and a false result of indeterminacy when the GBC and B are dropped from the model, again for both values of ξ.

6. This model is essentially the one studied by Mark Huggett (1993). It is discussed in some detail in chapter 5 of LS. Note that since there is no aggregate uncertainty in this economy, $R_t = R \quad \forall t$.
   (a) The household's Bellman equation is:

$$v(W,s) = \max_{c,W'} \left\{ u(c) + \beta \sum_{s'} \mathcal{P}(s,s') v(W',s') \right\}$$

   subject to

$$\begin{aligned} W' &= RW + y(s) - c \\ W &\geq -F \end{aligned}$$

   (b) For a given F, a stationary equilibrium is a fixed interest rate R, two policy functions, $W' = f(W,s)$ and $c = g(W,s)$, and a time-invariant distribution $\lambda(W,s)$ such that:
      (i) The policy functions solve the household's optimum problem.
      (ii) The distribution $\lambda(W,s)$ is stationary:

$$\lambda(W',s') = \sum_{s} \sum_{W \in \Omega(W',s)} \mathcal{P}(s,s') \lambda(W,s)$$

      where $\Omega(W',s) = \{W : W' = f(W,s)\}$.
      (iii) The loan market clears:

$$\sum_{W,s} \lambda(W,s) f(W,s) = 0.$$

      (iv) The goods market clears:

$$\sum_{W,s} \lambda(W,s) g(W,s) = \sum_{W,s} \lambda(W,s) y(s).$$

   (c) One algorithm to solve for this stationary equilibrium is
      (i) Guess an interest rate R.
      (ii) Solve the household's problem for the two policy functions $W' = f(W,s)$ and $c = g(W,s)$.
      (iii) Compute the associated stationary distribution $\lambda(W,s)$.
      (iv) Check to see if the loan market clears:

$$K = \sum_{W,s} \lambda(W,s) f(W,s).$$

      (v) If $K > 0$, go back to step 1 and try a smaller R. If $K < 0$, go back to step 1 and try a larger R. If $K = 0$, done.